In comparing (11.123) with atomic formula (11.121) we note that both V and V_N are attractive (although V_N is much larger), so that the signs of the spin-orbit energies are opposite. This means that in nuclei the single particle levels form "inverted" doublets. With a reasonable form for V_N, (11.123) is in qualitative agreement with the observed spin-orbit splittings in nuclei.*

The phenomenon of Thomas precession is presented from a more sophisticated point of view in Section 11.11 where the BMT equation is discussed.

11.9 Invariance of Electric Charge; Covariance of Electrodynamics

The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincaré before the formulation of the special theory of relativity. This invariance of form or *covariance* of the Maxwell and Lorentz force equations implies that the various quantities ρ, $\mathbf{J}$, $\mathbf{E}$, $\mathbf{B}$ that enter these equations transform in well-defined ways under Lorentz transformations. Then the terms of the equations can have consistent behavior under Lorentz transformations.

Consider first the Lorentz force equation for a particle of charge q,

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{11.124}$$

We know that $\mathbf{p}$ transforms as the space part of the 4-vector of energy and momentum,

$$p^\alpha = (p_0, \mathbf{p}) = m(U_0, \mathbf{U})$$

where $p_0 = E/c$ and U^α is the 4-velocity (11.36). If we use the proper time τ (11.26) instead of t for differentiation, (11.124) can be written

$$\frac{d\mathbf{p}}{d\tau} = \frac{q}{c}(U_0\mathbf{E} + \mathbf{U} \times \mathbf{B}) \tag{11.125}$$

The left-hand side is the space part of a 4-vector. The corresponding time component equation is the rate of change of energy of the particle (6.110):

$$\frac{dp_0}{d\tau} = \frac{q}{c}\mathbf{U}\cdot\mathbf{E} \tag{11.126}$$

If the force and energy change equations are to be Lorentz covariant, the right-hand sides must form the components of a 4-vector. They involve products of three factors, the charge q, the 4-velocity, and the electromagnetic fields. If the transformation properties of two of the three factors are known and Lorentz covariance is demanded, then the transformation properties of the third factor can be established.

Electric charge is absolutely conserved, as far as we know. Furthermore, the magnitudes of the charges of elementary particles (and therefore of any system

*See, for example, Section. 2.4c of A. Bohr and B. R. Mottelson, *Nuclear Structure*, Vol. 1, W. A. Benjamin, New York (1969).

of charges) are integral multiples of the charge of the proton. In the published literature,* it is experimentally established that the fractional difference between the magnitude of the electron's charge and the proton's charge is less than 10^{-19}, and unpublished results of King push this limit almost two orders of magnitude further.† The results of these experiments can be used to support the *invariance of electric charge* under Lorentz transformations or, more concretely, the independence of the observed charge of a particle on its speed. In his experiments King searched for a residual charge remaining in a container as hydrogen or helium gas is allowed to escape. No effect was observed and a limit of less than $10^{-19}e$ was established for the net charge per molecule for both H_2 and He. Since the electrons in He move at speeds twice as fast as in H_2, the charge of the electron cannot depend significantly on its speed, at least for speeds of the order of $(0.01–0.02)c$. In the experiment of Fraser, Carlson, and Hughes an atomic beam apparatus was used in an attempt to observe electrostatic deflection of beams of "neutral" cesium and potassium atoms. Again, no effect was observed, and a limit of less than 3.5×10^{-19} was set on the fractional difference between the charges of the proton and electron. Cesium and potassium have $Z = 55$ and 19, respectively. Thus the K-shell electrons in cesium at least move with speeds of order $0.4c$. The observed neutrality of the cesium atom at the level of 10^{-18}–10^{-19} is strong evidence for the invariance of electric charge.‡

The *experimental* invariance of electric charge and the requirement of Lorentz covariance of the Lorentz force equation (11.125) and (11.126) determines the Lorentz transformation properties of the electromagnetic field. For example, the requirement from (11.126) that $\mathbf{U} \cdot \mathbf{E}$ be the time component of a 4-vector establishes that the components of $\mathbf{E}$ are the time-space parts of a second rank tensor $F^{\alpha\beta}$, that is, $\mathbf{E} \cdot \mathbf{U} = F^{0\beta}U_\beta$. Although the explicit form of the field strength tensor $F^{\alpha\beta}$ can be found along these lines, we now proceed to examine the Maxwell equations themselves.

For simplicity, we consider the microscopic Maxwell equations, without $\mathbf{D}$ and $\mathbf{H}$. We begin with the charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{J}(\mathbf{x}, t)$ and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{11.127}$$

From the discussion at the end of Section 11.6 and especially (11.77) it is natural to postulate that ρ and $\mathbf{J}$ together form a 4-vector J^α:

$$J^\alpha = (c\rho, \mathbf{J}) \tag{11.128}$$

*J. G. King, *Phys. Rev. Lett.* **5**, 562 (1960); V. W. Hughes, L. J. Fraser, and E. R. Carlson, *Z. Phys. D-Atoms, Molecules and Clusters* **10**, 145 (1988). The latter tabulates many of the different methods and results.

†The limits on the measured charge per molecule in units of the electronic charge for H_2, He, and SF_6 were determined as 1.8 ± 5.4, -0.7 ± 4.7, 0 ± 4.3, respectively, all times 10^{-21}. Private communication from J. G. King (1975).

‡Mentioning only the electrons is somewhat misleading. The protons and neutrons inside nuclei move with speeds of the order $(0.2–0.3)c$. Thus the helium results of King already test the invariance of charge at appreciable speeds. Of course, if one is content with invariance at the level of 10^{-10} for $v/c \sim 10^{-3}$ the observed electrical neutrality of bulk matter when heated or cooled will suffice.

Then the continuity equation (11.127) takes the obviously covariant form,

$$\partial_\alpha J^\alpha = 0 \tag{11.129}$$

where the covariant differential operator ∂_α is given by (11.76). That J^α is a legitimate 4-vector follows from the invariance of electric charge: Consider a large number of elementary charges totaling δq at rest* in a small-volume element d^3x in frame K. They are idealized by a charge density ρ. The total charge $\delta q = \rho\, d^3x$ within the small-volume element is an experimental invariant; it is thus true that $\rho'\, d^3x' = \rho\, d^3x$. But the *four*-dimensional volume element $d^4x = dx^0\, d^3x$ is a Lorentz invariant:

$$d^4x' = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}\, d^4x = \det A\; d^4x = d^4x$$

The equality $\rho'\, d^3x' = \rho\, d^3x$ then implies that $c\rho$ transforms like x^0, namely, the time component of the 4-vector (11.128).

In the Lorenz family of gauges the wave equations for the vector potential $\mathbf{A}$ and the scalar potential Φ are

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= \frac{4\pi}{c}\mathbf{J} \\ \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \end{aligned} \tag{11.130}$$

with the Lorenz condition,

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{11.131}$$

The differential operator form in (11.130) is the invariant four-dimensional Laplacian (11.78), while the right-hand sides are the components of a 4-vector. Obviously, Lorentz covariance requires that the potentials Φ and $\mathbf{A}$ form a 4-vector potential,

$$A^\alpha = (\Phi, \mathbf{A}) \tag{11.132}$$

Then the wave equations and the Lorenz condition take on the manifestly covariant forms,

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

and

$$\partial_\alpha A^\alpha = 0 \tag{11.133}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of the potentials as

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{11.134}$$

*If there is a conduction current $\mathbf{J}$ as well as the charge density ρ in K, the total charge within d^3x is not an invariant. See *Møller*, Section 7.5. (His argument assumes the 4-vector character of $c\rho$ and $\mathbf{J}$, however.)

The x components of **E** and **B** are explicitly

$$\begin{aligned} E_x &= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0) \\ B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2) \end{aligned} \tag{11.135}$$

where the second forms follow from (11.132) and $\partial^\alpha = (\partial/\partial x_0, -\boldsymbol{\nabla})$. These equations imply that the electric and magnetic fields, six components in all, are the elements of a *second-rank, antisymmetric field-strength tensor*,

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{11.136}$$

Explicitly, the field-strength tensor is, in matrix form,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.137}$$

For reference, we record the field-strength tensor with two covariant indices,

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{11.138}$$

The elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to -\mathbf{E}$. Another useful quantity is the *dual field-strength tensor* $\mathscr{F}^{\alpha\beta}$. We first define the totally antisymmetric fourth-rank tensor $\epsilon^{\alpha\beta\gamma\delta}$:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for } \alpha = 0, \beta = 1, \gamma = 2, \delta = 3, \text{ and} \\ & \text{any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{11.139}$$

Note that the nonvanishing elements all have one time and three (different) space indices and that $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$. The tensor $\epsilon^{\alpha\beta\gamma\delta}$ is a *pseudotensor* under spatial inversions. This can be seen by contracting it with four different 4-vectors and examining the space inversion properties of the resultant rotationally invariant quantity. The dual field-strength tensor is defined by

$$\mathscr{F}^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{11.140}$$

The elements of the dual tensor $\mathscr{F}^{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$ in (11.137). This is a special case of the duality transformation (6.151).

To complete the demonstration of the covariance of electrodynamics we must write the Maxwell equations themselves in an explicitly covariant form. The inhomogeneous equations are

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$

In terms of $F^{\alpha\beta}$ and the 4-current J^α these take on the covariant form

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{11.141}$$

Similarly, the homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

can be written in terms of the dual field-strength tensor as

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \tag{11.142}$$

In terms of $F^{\alpha\beta}$, rather than $\mathscr{F}^{\alpha\beta}$, these homogeneous equations are the four equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{11.143}$$

where α, β, γ are any three of the integers 0, 1, 2, 3.

With the definitions of J^α (11.128), A^α (11.132), and $F^{\alpha\beta}$ (11.136), together with the wave equations (11.133) or the Maxwell equations (11.141) and (11.142), the covariance of the equations of electromagnetism is established. To complete the discussion, we put the Lorentz force and rate of change of energy equations (11.125) and (11.126) in manifestly covariant form,

$$\frac{dp^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta \tag{11.144}$$

The covariant description of the conservation laws of a combined system of electromagnetic fields and charged particles and a covariant solution for the fields of a moving charge are deferred to Chapter 12, where a Lagrangian formulation is presented.

For the macroscopic Maxwell equations it is necessary to distinguish two field-strength tensors, $F^{\alpha\beta} = (\mathbf{E}, \mathbf{B})$ and $G^{\alpha\beta} = (\mathbf{D}, \mathbf{H})$, where $F^{\alpha\beta}$ is given by (11.137) and $G^{\alpha\beta}$ is obtained from (11.137) by substituting $\mathbf{E} \rightarrow \mathbf{D}$ and $\mathbf{B} \rightarrow \mathbf{H}$. The covariant form of the Maxwell equations is then

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \qquad \partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \tag{11.145}$$

It is clear that with the fields (**E**, **B**) and (**D**, **H**) transforming as antisymmetric second-rank tensors the polarization **P** and the negative magnetization $-\mathbf{M}$ form a similar second-rank tensor. With these quantities given meaning as macroscopic averages of atomic properties *in the rest frame* of the medium, the electrodynamics of macroscopic matter in motion is specified. This is the basis of the electrodynamics of Minkowski and others. For further information on this rather large and important subject, the reader can consult the literature cited at the end of the chapter.

11.10 *Transformation of Electromagnetic Fields*

Since the fields **E** and **B** are the elements of a second-rank tensor $F^{\alpha\beta}$, their values in one inertial frame K' can be expressed in terms of the values in another inertial frame K according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{11.146}$$

In the matrix notation of Section 11.7 this can be written

$$F' = AF\tilde{A} \tag{11.147}$$

where F and F' are 4×4 *matrices* (11.137) and A is the Lorentz transformation matrix of (11.93). For the specific Lorentz transformation (11.95), corresponding to a boost along the x_1 axis with speed $c\beta$ from the unprimed frame to the primed frame, the explicit equations of transformation are

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned} \tag{11.148}$$

Here and below, the subscripts 1, 2, 3 indicate ordinary Cartesian spatial components and are not covariant indices. The inverse of (11.148) is found, as usual, by interchanging primed and unprimed quantities and putting $\beta \to -\beta$. For a general Lorentz transformation from K to a system K' moving with velocity $\mathbf{v}$ relative to K, the transformation of the fields can be written

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \end{aligned} \tag{11.149}$$

These are the analogs for the fields of (11.19) for the coordinates. Transformation (11.149) shows that **E** and **B** have no independent existence. A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame. Of course certain restrictions apply (see Problem 11.14) so that, for example, a purely electrostatic field in one coordinate system cannot be transformed into a purely magnetostatic field in another. But the fields are completely interrelated, and one should properly speak of the electromagnetic field $F^{\alpha\beta}$, rather than **E** or **B** separately.

If no magnetic field exists in a certain frame K', as for example with one or more point charges at rest in K', the inverse of (11.149) shows that in the frame K the magnetic field **B** and electric field **E** are linked by the simple relation

$$\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E} \tag{11.150}$$